

A WEIGHT MULTIPLICITY FORMULA FOR DEMAZURE MODULES

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ABSTRACT. We establish a formula for the weight multiplicities of Demazure modules (in particular for highest weight representations) of a complex connected algebraic group in terms of the geometry of its Langlands dual.

INTRODUCTION

Let G be a complex, connected, simple algebraic group and let B and T be a Borel subgroup and maximal torus of G contained in B . For an integral dominant weight λ , denote by V_λ the integrable highest weight G -module with highest weight λ . The Demazure modules $D_{w(\lambda)}$ (with w running over the Weyl group of G) form a filtration of V_λ . The module $D_{w(\lambda)}$ is defined as the B -submodule of V_λ generated by the extremal weight space of weight $w(\lambda)$ (which are one-dimensional). In particular, if w_\circ is the longest element in the Weyl group of G , $D_{w_\circ(\lambda)}$ is precisely V_λ . More importantly, the Demazure modules can be realized as spaces of global sections of certain line bundles over Schubert varieties (see Section 2 for details).

The main goal of this note is to explain how the recent developments [5], [6] in the theory of nonsymmetric Macdonald polynomials have as consequence a geometric formula for weight multiplicities in Demazure modules and, in particular, irreducible G -modules. We will briefly state our result.

Let G^\vee be the unique complex, connected, reductive group whose root datum is dual to that of G . Let us also consider B^\vee , a Borel subgroup of G^\vee . The group G^\vee is also defined over $\mathcal{O} := \mathbb{C}[[x]]$. We denote $G^\vee(\mathcal{O})$ by K and let I be the subgroup of K defined as the inverse image of $B^\vee(\mathbb{C})$ under the reduction map $K \rightarrow G^\vee(\mathbb{C})$. The space $G^\vee(\mathbb{C}((x)))/I$ is endowed with a structure of ind-variety over \mathbb{C} . For any integral weights λ and μ let $M_{\lambda,\mu}(\mathbb{C})$ be the finite dimensional subvariety of $G^\vee(\mathbb{C}((x)))/I$ defined in Section 4.1.

Theorem 1. *Let λ and μ be two integral weights such that $\mu \leq \lambda$. The weight multiplicity $m_{\lambda,\mu}$ of the weight λ in the Demazure module D_λ equals the number of top dimensional irreducible components of $M_{\lambda,\mu}(\mathbb{C})$.*

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Other geometric formulas for weight multiplicities in irreducible G -modules exist in the literature [8], [9] but no such formula was available for Demazure modules. Our result is similar to one consequence [9, Corollary 7.4] of the recent work of Mirković and Vilonen on the geometric Satake isomorphism (see 4.3 on further comments about this connection). However, we should keep in mind that the Demazure modules are generally just B -modules and their result covers only the case of reductive groups; it would be interesting to investigate if the ideas in [9] could be used to cover a larger class of groups.

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1. PRELIMINARIES

1.1. Let \mathfrak{g} be the complex, simple Lie algebra of rank n and let \mathfrak{h} and \mathfrak{b} be a Cartan subalgebra and, respectively, a Borel subalgebra containing \mathfrak{h} . Let \check{R} be root system of \mathfrak{g} with respect to \mathfrak{h} and denote by R^+ the set of roots of \mathfrak{b} with respect to \mathfrak{h} . The set of positive simple roots is denoted by $\{\alpha_1, \dots, \alpha_n\}$ and $\{\lambda_1, \dots, \lambda_n\}$ will denote the fundamental weights. As usual, \check{Q} denotes the root lattice of \check{R} and P denotes its weight lattice.

The vector space $\mathfrak{h}_{\mathbb{R}}^*$ (the real vector space spanned by the roots) has a canonical scalar product (\cdot, \cdot) which we normalize such that it gives square length 2 to the short roots in \check{R} (if there is only one root length we consider all roots to be short). We will use \check{R}_s and \check{R}_ℓ to refer to the short and respectively long roots in \check{R} . We will identify the vector space $\mathfrak{h}_{\mathbb{R}}$ (the real vector space spanned by the coroots) with its dual using this scalar product. Under this identification $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for any root α . The root θ is defined as the *highest short root* in \check{R} . Also, let us consider

$$\rho = \frac{1}{2} \sum_{\alpha \in R} \alpha^\vee$$

To the finite root system \check{R} we will associate an *affine root system* R . Let $\text{Aff}(\mathfrak{h}_{\mathbb{R}})$ be the space of affine linear transformations of $\mathfrak{h}_{\mathbb{R}}$. As a vector space, it can be identified to $\mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$ via

$$(f + c\delta)(x) = f(x) + c, \quad \text{for } f \in \mathfrak{h}_{\mathbb{R}}^*, x \in \mathfrak{h}_{\mathbb{R}} \text{ and } c \in \mathbb{R}$$

Let r denote the maximal number of laces connecting two vertices in the Dynkin diagram of \check{R} . Then,

$$R := (\check{R}_s + \mathbb{Z}\delta) \cup (\check{R}_\ell + r\mathbb{Z}\delta) \subset \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta$$

The set of affine positive roots R^+ consists of affine roots of the form $\alpha + k\delta$ such that k is non-negative if α is a positive root, and k is strictly positive if α is a negative root. The affine simple roots are $\{\alpha_i\}_{0 \leq i \leq n}$, where we set $\alpha_0 := \delta - \theta$.

1.2. The scalar product on $\mathfrak{h}_{\mathbb{R}}^*$ can be extended to a non-degenerate bilinear form on the real vector space

$$H := \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$$

by requiring that $(\delta, \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\delta) = (\Lambda_0, \mathfrak{h}_{\mathbb{R}}^* \oplus \mathbb{R}\Lambda_0) = 0$ and $(\delta, \Lambda_0) = 1$. Given $\alpha \in R$ and $x \in H$ let

$$s_{\alpha}(x) := x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$$

The *affine Weyl group* W is the subgroup of $\text{GL}(H)$ generated by all s_{α} (the simple reflections $s_i = s_{\alpha_i}$ are enough). The *finite Weyl group* \check{W} is the subgroup generated by s_1, \dots, s_n . It is easy to see that $\mathfrak{h}_{\mathbb{R}}^* + \mathbb{R}\delta + \Lambda_0$ is stable under the action of W . Therefore, if we identify $\mathfrak{h}_{\mathbb{R}}^*$ with $\mathfrak{h}_{\mathbb{R}}^* + \mathbb{R}\delta + \Lambda_0$ we obtain an affine action of W on $\mathfrak{h}_{\mathbb{R}}^*$; we denote by $w \cdot x$ the affine action of $w \in W$ on $x \in \mathfrak{h}_{\mathbb{R}}^*$. For example, the affine action of s_0 can be described as follows

$$s_0 \cdot x = s_{\theta}(x) + \theta \quad (1)$$

We define the fundamental affine chamber as

$$\mathcal{C} := \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid (x + \Lambda_0, \alpha_i^{\vee}) \geq 0, \ 0 \leq i \leq n\} \quad (2)$$

The non-zero elements of $\mathcal{O}_P := P \cap \mathcal{C}$ are the so-called minuscule weights. Let us remark that each orbit of the affine action of W on P contains the origin or a unique element of \mathcal{O}_P .

1.3. For each w in W let $\ell(w)$ be the length of a reduced (i.e. shortest) decomposition of w in terms of simple reflections. The length of w can be also geometrically described as follows. For any affine root α , denote by H_{α} the affine hyperplane consisting of fixed points of the affine action of s_{α} on $\mathfrak{h}_{\mathbb{R}}^*$. Then, $\ell(w)$ equals the number of affine hyperplanes H_{α} separating \mathcal{C} and $w \cdot \mathcal{C}$. For any affine transformation of $\mathfrak{h}_{\mathbb{R}}^*$ which preserves the set of hyperplanes $\{H_{\alpha}\}_{\alpha \in R}$, we can use the geometric point of view to define the length of that transformation. For example, for a weight λ we define the following affine transformation of $\mathfrak{h}_{\mathbb{R}}^*$

$$\tau_{\lambda}(x) = x + \lambda$$

It is easy to check that τ_{λ} has the required properties to allow us to talk about its length. In fact a concrete formula for its length is available (see, for example, [5, (5)])

$$\ell(\tau_{\lambda}) = \sum_{\alpha \in \check{R}} |(\lambda, \alpha^{\vee})| \quad (3)$$

For each weight λ define λ_- , respectively $\tilde{\lambda}$, to be the unique element in $\check{W}(\lambda)$, respectively $W \cdot \lambda$, which is an anti-dominant weight, respectively an element of \mathcal{O}_P (that is a minuscule weight or zero), and $\check{w}_{\lambda}^{-1} \in \check{W}$, respectively $w_{\lambda}^{-1} \in W$, to be the unique minimal length element by which this is achieved. Also, for each weight

λ define λ_+ to be the unique element in $\mathring{W}(\lambda)$ which is dominant and denote by w_\circ the maximal length element in \mathring{W} . It was shown in [5, Lemma 1.7 (3)] that the following equality holds for any weight λ

$$\ell(\tau_\lambda) = \ell(w_\lambda) + \ell(\mathring{w}_\lambda) \quad (4)$$

For later use we record the following.

Lemma 1.1. *Let λ be a weight. Then*

$$2\langle \lambda, \rho \rangle \leq \ell(w_\lambda) + \ell(\mathring{w}_\lambda)$$

Proof. From (3) we know that $2\langle \lambda, \rho \rangle \leq \ell(\tau_\lambda)$. Now, (4) implies the desired result. \square

1.4. The Bruhat order is a partial order on any Coxeter group defined in way compatible with the length function. For an element w we put $w < s_i w$ if and only if $\ell(w) < \ell(s_i w)$. The transitive closure of this relation is called the Bruhat order. The terminology is motivated by the way this ordering arises for Weyl groups in connection with inclusions among closures of Bruhat cells for a corresponding semisimple algebraic group.

We can use the Bruhat order on W to define a partial order on the weight lattice which will also be called the Bruhat order. For any $\lambda, \mu \in P$ we write

$$\lambda < \mu \text{ if and only if } w_\lambda < w_\mu \quad (5)$$

The minimal elements of the weight lattice with respect to this partial order are the minuscule weights and if $\lambda < \mu$ then necessarily $\tilde{\lambda} = \tilde{\mu}$.

Lemma 1.2. *Let λ and μ be two weights such that $\mu \leq \lambda$. Define the rational number*

$$n_{\lambda, \mu} := \frac{1}{2}\ell(w_\lambda) - \frac{1}{2}\ell(\mathring{w}_\lambda) + \ell(w_\circ) - \langle \mu, \rho \rangle \quad (6)$$

Then $n_{\lambda, \mu}$ is a positive integer.

Proof. Let us argue first that $n_{\lambda, \mu}$ is positive. We can use Lemma 1.1 to obtain

$$n_{\lambda, \mu} \geq \frac{1}{2}\ell(w_\lambda) - \frac{1}{2}\ell(\mathring{w}_\lambda) + \ell(w_\circ) - \frac{1}{2}\ell(w_\mu) - \frac{1}{2}\ell(\mathring{w}_\mu)$$

Keeping in mind that $\ell(w_\lambda) \geq \ell(w_\mu)$ (which is a consequence of the hypothesis) and that w_\circ is the maximal length element in \mathring{W} , our claim immediately follows.

To show that $n_{\lambda, \mu}$ is integer, let us remark that it is enough to check that $\frac{1}{2}\ell(w_\lambda) + \frac{1}{2}\ell(\mathring{w}_\lambda)$ is integer. This fact follows from (4) together with

$$\ell(\tau_\lambda) = \ell(\tau_{\lambda_+}) = 2\langle \lambda_+, \rho \rangle$$

\square

2. DEMAZURE MODULES

Let G be a complex, connected, simple algebraic group with Lie algebra \mathfrak{g} and denote by T and B the maximal torus and the Borel subgroup of G with Lie algebras \mathfrak{h} and \mathfrak{b} , respectively.

For an integral dominant weight λ , denote by V_λ the integrable highest weight \mathfrak{g} -module with highest weight λ . It is well known that V_λ is an irreducible \mathfrak{g} -module. Hence, for any w in \check{W} the $w(\lambda)$ -weight space $V_{\lambda, w(\lambda)}$ is one-dimensional. The Demazure module $D_{w(\lambda)}$ is defined as the \mathfrak{b} -module generated by $V_{\lambda, w(\lambda)}$. Since λ is integral $D_{w(\lambda)}$ is also a B -module. The Demazure modules associated to a fixed integral dominant weight λ form a filtration (with respect to the Bruhat order on \check{W}) of V_λ . In particular, $D_\lambda = V_{\lambda, \lambda}$ and $D_{w_\circ(\lambda)} = V_\lambda$.

There exists an important geometrical construction of Demazure modules which relates them with the geometry of Schubert varieties in the flag variety G/B . For any w in \check{W} let

$$X_w = \overline{BwB/B} \subseteq G/B$$

denote the Schubert variety associated to w . The Schubert varieties are closed finite dimensional projective irreducible subvarieties of G/B . For any integral weight λ we denote by e^λ the character of B obtained from T via the isomorphism $T \simeq B/[B, B]$. Consider the fiber product $\mathcal{L}_\lambda = G \times_B \mathbb{C}_\lambda$, where \mathbb{C}_λ denotes \mathbb{C} equipped with the B -action given by the character e^λ . The natural projection $G \times \mathbb{C}_\lambda \rightarrow G$ induces a well defined G -equivariant holomorphic map $\mathcal{L}_\lambda \rightarrow G/B$; in other words \mathcal{L}_λ becomes a G -equivariant holomorphic line bundle over G/B . By restriction we obtain a line bundle $\mathcal{L}_{\lambda, w}$ over X_w . Since X_w is B -invariant, the space of holomorphic sections $H^0(X_w, \mathcal{L}_{\lambda, w})$ admits a B -module structure. The relation with the Demazure module $D_{w(\lambda)}$ is the following.

Theorem 2.1. *Let λ be a dominant integral weight and let w be an element of W . Then, $D_{w(\lambda)}$ and $H^0(X_w, \mathcal{L}_{-\lambda, w})^*$ are isomorphic as B -modules.*

The result holds in the more general setting of generalized flag varieties of Kac-Moody groups. For a proof see, for example, Corollary 8.1.26 in [7].

Let λ be an arbitrary integral weight. As $T \subset B$, the Demazure module D_λ is also a T -module. Its T -character will be denoted by χ_λ . Let us write

$$\chi_\lambda = \sum_{\mu \in X^*(T)} m_{\lambda, \mu} e^\mu \tag{7}$$

where we denoted by $X^*(T)$ the character group of the torus T . The non-negative integers $m_{\lambda, \mu}$ appearing in the above formula are the multiplicities of the weights μ in the Demazure module D_λ . As remarked before, if λ is dominant, χ_λ equals e^λ and $\chi_{w_\circ(\lambda)}$ is the character of V_λ .

3. NONSYMMETRIC MACDONALD POLYNOMIALS

3.1. Let us introduce a field $\mathbb{C}_{q,t}$ (of parameters) as follows. Let q and t be two formal parameters and let m be the lowest common denominator of the rational numbers $\{(\alpha_j, \lambda_k) \mid 1 \leq j, k \leq n\}$. The field $\mathbb{C}_{q,t}$ is defined as the field of rational functions in $q^{\frac{1}{m}}$ and $t^{\frac{1}{2}}$ with complex coefficients. We will also use the field of rational functions in $t^{\frac{1}{2}}$ denoted by \mathbb{C}_t . The algebra $\mathcal{R}_{q,t} = \mathbb{C}_{q,t}[e^\lambda; \lambda \in P]$ is the group $\mathbb{C}_{q,t}$ -algebra of the lattice P . Similarly, the algebra $\mathcal{R}_t = \mathbb{C}_t[e^\lambda; \lambda \in P]$ is the group \mathbb{C}_t -algebra of the lattice P .

The nonsymmetric Macdonald polynomials $E_\lambda(q, t)$ associated to the root system \tilde{R} are remarkable family of polynomials indexed by the weight lattice P and which form a basis for $\mathcal{R}_{q,t}$. They were defined by Opdam and Macdonald for various specializations of the parameters and in full generality by Cherednik [1]. We refer the reader to [1] for a detailed account of their construction and basic properties. We only mention at this point that they satisfy the following triangularity property with respect to the Bruhat order

$$E_\lambda(q, t) \in \text{span}\langle e^\mu \mid \mu \leq \lambda \rangle$$

3.2. Although, a priori, the coefficients of the polynomials $E_\lambda(q, t)$ are just rational functions in q and t it was proved in [5, Section 3.1] that in fact each coefficient has a finite limit as q approaches infinity and, in consequence,

$$E_\lambda(t) = \lim_{q \rightarrow \infty} E_\lambda(q, t)$$

is well defined as an element of \mathcal{R}_t .

Another important fact is that the coefficients of $E_\lambda(t)$ are polynomials in t^{-1} and in consequence their limit as t approaches infinity is finite. Moreover, the following is true. Let G , B and T be a complex connected simple algebraic group, a Borel subgroup and a maximal torus as in Section 2. We can certainly regard $X^*(T)$, the character group of the torus T , as being a sublattice of P . Therefore, the Demazure character χ_λ associated to an integral weight λ becomes an element of \mathcal{R}_t , and the formula (7) will be regarded as an identity in \mathcal{R}_t . The following result was proved in [5, Corollary 3.8].

Theorem 3.1. *Let λ be an arbitrary integral weight for G . Then,*

$$\chi_\lambda = \lim_{t \rightarrow \infty} E_\lambda(t) \tag{8}$$

3.3. For special values of the parameter t the coefficients of the polynomials $E_\lambda(t)$ have a rather different interpretation. To be able to state the result we first need to introduce more notation.

Given λ in P let us define the following normalization factor

$$j_\lambda = t^{\frac{1}{2}(\ell(w_\lambda) - \ell(\hat{w}_\lambda))} \quad (9)$$

where w_λ and \hat{w}_λ are the Weyl group elements defined in Section 1. Let us remark that, as it follows from [5, Corollary 3.4], the normalization factor denoted by the same symbol in [6, (12)] is, for equal values of the parameters, precisely the element defined above.

3.4. Let \mathbf{G}^\vee be the unique split, connected, reductive group scheme whose root datum is dual to that of G (the Chevalley group scheme). Let \mathbf{B}^\vee be a Borel subgroup of \mathbf{G}^\vee and \mathbf{T}^\vee a maximal split torus of \mathbf{G}^\vee contained in \mathbf{B}^\vee . The Borel \mathbf{B}^\vee has the Levi decomposition $\mathbf{B} = \mathbf{T}^\vee \mathbf{U}$, where \mathbf{U} is the unipotent radical of \mathbf{B}^\vee . The unique Borel subgroup of \mathbf{G}^\vee which is opposed to \mathbf{B}^\vee with respect to \mathbf{T}^\vee will be denoted by $\overline{\mathbf{B}}^\vee$ and $\overline{\mathbf{U}}$ denotes its unipotent radical. We have deliberately ignored the field of definition since all above groups are defined over \mathbb{Z} (the structure constants involve only integers).

Let p be a prime number and \mathfrak{t} a positive integer power of p . We will denote by $\mathbb{F}_\mathfrak{t}$ the finite field with \mathfrak{t} elements. For the moment let k denote an arbitrary field. Let x be a formal parameter and let $F := k((x))$ be the quotient field of $\mathcal{O} := k[[x]]$ (formal power series with coefficients in k). Of course, F is a p -adic field, \mathcal{O} is its ring of integers and k is the residue field.

The F -rational points of \mathbf{G}^\vee will be denoted by $G^\vee(F)$ (or simply by G^\vee if the field F is implicitly understood) and the same type of notation will be used for all the linear algebraic groups defined above. Each $\lambda \in X^*(T)$ becomes an element of $X_*(T^\vee)$ (the cocharacter group of T^\vee) and therefore determines a morphism $\lambda : F^\times \rightarrow T^\vee$. We will denote by x^λ the image of x under the above morphism. The groups \mathbf{G}^\vee , \mathbf{T}^\vee , \mathbf{U} , $\overline{\mathbf{U}}$ are also defined over \mathcal{O} . We will denote $G^\vee(\mathcal{O})$ by K and denote by I the Iwahori subgroup defined as the inverse image of $B^\vee(k)$ under $G^\vee(\mathcal{O}) \rightarrow G^\vee(k)$. Of course, all the above considerations hold if we replace k by \mathbb{Z} .

3.5. In this subsection we will assume that the field k is $\mathbb{F}_\mathfrak{t}$. Let us remark that the affine root system associated to (G^\vee, T^\vee) is in fact R^\vee and that K is a maximal compact subgroup of G^\vee .

The group G^\vee is unimodular as opposed to the Borel subgroup B^\vee which is not. We choose a Haar measure on G^\vee normalized such that the Iwahori subgroup I has volume one. The modular function of the Borel subgroup $\delta : T^\vee \rightarrow \mathbb{R}_+^\times$ is defined by $\delta(a) = |\det(\text{Ad}|_U(a))|$ for any a in T^\vee ; we denoted by $\text{Ad}|_U$ the automorphism of the Lie algebra of U given by the adjoint representation and we denoted by $|\cdot|$ the usual metric on F induced by the valuation. Since G^\vee is split, the formula for

the modular function takes the following form on the elements x^λ , for $\lambda \in X^*(T)$

$$\delta(x^\lambda) = \mathfrak{t}^{2\langle \lambda, \rho \rangle} \quad (10)$$

For an element $f = f(t)$ in \mathcal{R}_t we will write $f(\mathfrak{t})$ to refer to the element of $\mathbb{C}[e^\lambda; \lambda \in P]$ obtained by substituting the positive integer \mathfrak{t} for the parameter t . We are now ready to state a result connecting the coefficients of nonsymmetric Macdonald polynomials with the geometry of the group G^\vee . The Theorem stated below was proved in [6] (see Theorem 5.10 and formula (25)).

Theorem 3.2. *Let λ be an element of $X^*(T)$ (or, equivalently, of $X_*(T^\vee)$). The coefficients appearing in*

$$E_\lambda(\mathfrak{t}) = \sum_{\mu \leq \lambda} c_{\lambda, \mu} e^\mu$$

are given by

$$c_{\lambda, \mu} = \frac{\text{vol}(\overline{U}x^{-\mu}I \cap Kx^{-\lambda}I)}{j_\lambda(\mathfrak{t})\mathfrak{t}^{\ell(w_\circ)}\delta^{\frac{1}{2}}(x^{w_\circ(\mu)})}$$

We would prefer to make the denominator in the above formula as explicit as possible.

Lemma 3.3. *Let λ and μ be two elements of $X^*(T)$ for which $\mu \leq \lambda$. Then,*

$$j_\lambda(\mathfrak{t})\mathfrak{t}^{\ell(w_\circ)}\delta^{\frac{1}{2}}(x^{w_\circ(\mu)}) = \mathfrak{t}^{n_{\lambda, \mu}}$$

Proof. Straightforward from (6), (9) and (10). \square

By combining Theorem 3.1 and Theorem 3.2 we can see that the weight multiplicities (as defined by (7)) of the Demazure module D_λ can be computed as follows.

Corollary 3.4. *Let λ and μ be two elements of $X^*(T)$ for which $\mu \leq \lambda$. Then,*

$$m_{\lambda, \mu} = \lim_{\mathfrak{t} \rightarrow \infty} \frac{\text{vol}(\overline{U}x^{-\mu}I \cap Kx^{-\lambda}I)}{\mathfrak{t}^{n_{\lambda, \mu}}} \quad (11)$$

4. THE MULTIPLICITY FORMULA

4.1. The varieties $M_{\lambda, \mu}$. Let us assume for a moment that $k = \mathbb{C}$. It is well-known (see, for example, [9] and the references therein) that the space G^\vee/K is an *ind-variety* (i.e. admits an increasing filtration with varieties such all inclusion maps among them are closed embeddings) defined over \mathbb{C} . We refer the reader to [7, Chapter IV] for a brief introduction to ind-varieties. In our case, the members of the filtration on G^\vee/K can be constructed from a filtration of G^\vee obtained by bounding the number of poles with respect to x of the matrix coefficients for a faithful representation of G^\vee .

The space G^\vee/K is usually referred to as the *affine Grassmannian* of $G^\vee(\mathbb{C})$. However, because G^\vee and K are still defined for $k = \mathbb{Z}$, the same geometric constructions go through in this case, and G^\vee/K acquires a structure of ind-variety defined over \mathbb{Z} (see also the remarks in Section 14 of [9]). In a completely similar way, keeping in mind that I is defined for $k = \mathbb{Z}$, we endow G^\vee/I with a structure of ind-variety defined over \mathbb{Z} .

Let us consider the map $\pi : G^\vee \rightarrow G^\vee/I$. The group K acts on G^\vee/I by finite dimensional orbits. Indeed, the orbits are all of the form $\pi(Kx^\nu I)$, and the number of poles of the elements in $Kx^\nu I$ is bounded by

$$\max_{1 \leq i \leq n} |\langle \nu, \alpha_i^\vee \rangle|$$

Therefore, $\pi(Kx^\nu I)$ is included in one member of the filtration on G^\vee/I and hence it is finite dimensional. For any λ and μ in $X_*(T)$, let us define the variety $M_{\lambda,\mu}$ as $\pi(\overline{U}x^{-\mu}I \cap Kx^{-\lambda}I)$. From the above considerations it is clear that the varieties $M_{\lambda,\mu}$ are finite dimensional and defined over \mathbb{Z} .

4.2. Proof of Theorem 1. We will show that the right hand side of the formula (11) equals the number of irreducible components of top dimension of $M_{\lambda,\mu}(\mathbb{C})$. Let us recall that Corollary 3.4 holds under the hypothesis $k = \mathbb{F}_t$. Because the Iwahori subgroup I has volume one we can regard the volume of the set $\overline{U}x^{-\mu}I \cap Kx^{-\lambda}I$ as the number of right I -cosets in $\overline{U}x^{-\mu}I \cap Kx^{-\lambda}I$ or, equivalently, as the number of points in $M_{\lambda,\mu}(\mathbb{F}_t)$ (which we denote by $|M_{\lambda,\mu}(\mathbb{F}_t)|$).

The Lefschetz fixed point formula (for the Frobenius automorphism) and Theorem 1 in [2] (for $\mathcal{F}_0 = \mathbb{Q}_\ell$, $S_0 = \text{Spec}(\mathbb{F}_t)$ and $X_0 = M_{\lambda,\mu}$) let us conclude that

$$\lim_{t \rightarrow \infty} \frac{|M_{\lambda,\mu}(\mathbb{F}_t)|}{t^{n_{\lambda,\mu}}}$$

is indeed the number of irreducible components of top dimension of $M_{\lambda,\mu}(\mathbb{C})$.

As an immediate consequence we obtain the following.

Corollary 4.1. *Let λ and μ be two elements of $X^*(T)$ for which $\mu \leq \lambda$. The dimension of $M_{\lambda,\mu}(\mathbb{C})$ is $n_{\lambda,\mu}$.*

4.3. Final remarks. If we keep in mind that for anti-dominant λ the Demazure module D_λ is in fact V_{λ_+} , the irreducible G -module with highest weight $\lambda_+ = w_\circ(\lambda)$ (or lowest weight λ), our result gives a geometric formula for the weight multiplicities of V_{λ_+} . However, if we would be only interested in a formula for weight multiplicities in V_λ (here λ is dominant), this could be obtained by closely following the argument given above but replacing in Theorem 3.1 the nonsymmetric Macdonald polynomials with symmetric Macdonald polynomials $P_\lambda(t)$ (at $q \rightarrow \infty$) and Theorem 3.2 with corresponding result stating the fact that $P_\lambda(t)$ arises as the

Satake transform of the characteristic function of the set $Kx^\lambda K$. If we denote by $Z_{\lambda,\mu}(\mathbb{C})$ the \mathbb{C} -points of the image of $Ux^\mu K \cap Kx^\lambda K$ through the map

$$\tilde{\pi} : G^\vee \rightarrow G^\vee / K$$

the corresponding result read as follows.

Theorem 4.2. *Let λ and μ be two dominant integral weights such that $\mu \leq \lambda$. The weight multiplicity $m_{\lambda,\mu}$ of the weight μ in the irreducible highest weight module V_λ equals the number of top dimensional irreducible components of $Z_{\lambda,\mu}(\mathbb{C})$.*

The above statement is quite close to one of results in [9] which we will briefly recall. For the following statements we assume that $k = \mathbb{C}$. First, Theorem 3.2 in [9] shows that $\tilde{\pi}(Ux^\mu K) \cap \overline{\tilde{\pi}(Kx^\lambda K)}$ (the closure is in G^\vee / K) is pure dimensional (which, of course, implies that $Z_{\lambda,\mu}$ is pure dimensional). Second, as a consequence of the equivalence of tensor categories between the category of representations of G and the category of K -equivariant perverse sheaves on the affine Grassmannian G^\vee / K with \mathbb{C} -coefficients, Corollary 7.4 in [9] states that the weight multiplicity $m_{\lambda,\mu}$ equals the number of irreducible components of $\tilde{\pi}(Ux^\mu K) \cap \overline{\tilde{\pi}(Kx^\lambda K)}$. The absence of the closure in Theorem 4.2 slightly improves this statement.

The main result in [9] (the above mentioned equivalence of categories) was proved for reductive groups. Keeping in mind that Theorem 1 holds for Demazure modules (which are just B -modules) and that B is a solvable group, our result seems to suggest that the results of Mirković and Vilonen might eventually be extended to a larger class of groups and representations.

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